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Large-N Weinberg-Tomozawa interaction and spin-flavor symmetry

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Abstract. The construction of an extended version of the Weinberg-Tomozawa Lagrangian, in which baryons and mesons form spin-flavor multiplets, is reviewed and some of its properties discussed, for an arbitrary number of colors and flavors. The coefficient tables of spin-flavor irreducible representations related by crossing between the s-, t- and u-channels are explicitly constructed.

1 Introduction

Spin-flavor symmetry, the symmetry by which $SU(N_f)$ is promoted to $SU(2N_f)$, was fashionable in the early days of the quark model [1] due to its predictive power and successes in the description of hadronic properties, but was largely left aside in favor of QCD when the latter theory was developed. In addition symmetries including together Poincaré and internal groups were shown to be of necessity of approximated type [2]. As it turned out, large-N QCD (one of the few techniques available to attack QCD in the non perturbative regime) was shown to display spin-flavor symmetry for the lightest baryon of the theory [3].

The success of chiral unitarization schemes in the study of meson-baryon resonances within a Bethe-Salpeter approach [4], indicated the opportunity of considering their large-N versions. In particular, in the s-wave 0^- meson–$1/2^+$ baryon sector, appropriate to investigate negative-parity baryons, a large simplification of the formalism was obtained if the introduction of spin-flavor multiplets for baryon was complemented with the similar approach in the meson sector, thereby including vector mesons in the game [5]. This naturally leads to an extended version of the s-wave Weinberg-Tomozawa (WT) Lagrangian [6] with spin-flavor symmetry, the construction of which is reviewed below, in the first part of the paper. This interaction turns out to have a very constrained form which follows entirely from chiral symmetry considerations. Chiral symmetry consistent with spin-flavor has been advocated by Caldi and Pagels [7]. In the second part of the paper we present new and detailed results concerning crossing coefficients. They describe how interactions driven by specific $SU(2N_f)$ irreducible representations in the t- or u-channels are decomposed in the s-channel.

2 WT interaction and spin-flavor symmetry

In what follows N denotes the number of colors, N_f the number of flavors and $n = 2N_f$. In the large-N limit, the spin-flavor $SU(n)$ symmetry becomes exact for the QCD baryon [3]. The baryon is described by a field $\mathcal{B}^{i_1 \cdots i_N}(x)$. The label i contains spin and flavor and runs from 1 to n. This field is a fully symmetric $SU(n)$ tensor, *i.e.*, falls in the representation with Young tableau $[N]$. (The fully antisymmetric color factor of the wave function is implicit.) For $N = N_f = 3$ this is the irrep **56** of $SU(6)$, so for general N and N_f it is sometimes denoted by "**56**". Assuming spin-flavor symmetry in the mesonic sector (not a direct consequence of large N QCD), the lightest mesons fall in the adjoint representation of $SU(n)$, "**35**" with tableau $[2, 1^{n-2}]$, with linear field $\Phi^i{}_j(x)$ and non-linear field $\mathcal{U} = \exp(2i\Phi/f)$.

The baryon and linear meson fields transform and are normalized as $Q^{i_1} \cdots Q^{i_N}$ and $Q^i \overline{Q}_j - (1/n) Q^k \overline{Q}_k \delta^i{}_j$, respectively, where Q^i and $\overline{Q}_i$ are quark and the antiquark fields,

$$Q^i \to U^i{}_j Q^j, \quad \overline{Q}_i \to (U^i{}_j)^* \overline{Q}_j = \overline{Q}_j (U^{-1})^j{}_i, \ U \in SU(n). \tag{1}$$

As in the $SU(3)$ case, the WT Lagrangian is obtained from the free baryonic Lagrangian by replacing the derivative by a covariant derivative which includes the

mesons

$$\mathcal{L} = \frac{1}{N!} \mathcal{B}^{\dagger}_{i_1 \cdots i_N} \left(i\nabla_0 - M + \frac{1}{2M} \boldsymbol{\nabla}^2 \right) \mathcal{B}^{i_1 \cdots i_N} . \qquad (2)$$

The covariant derivative $\nabla_\mu = \partial_\mu + A_\mu$ has connection

$$A_\mu(x) = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)$$

$$= \frac{1}{2f^2}[\Phi, \partial_\mu \Phi] + \mathcal{O}(\Phi^3) \qquad (3)$$

(where $u^2 = \mathcal{U}$). This yields a spin-flavor extended WT interaction

$$\mathcal{L}^{\mathrm{sf}}_{\mathrm{WT}} = \frac{iN}{2f^2}[\Phi, \partial_\mu \Phi]^j{}_k \frac{1}{N!} \mathcal{B}^{\dagger}_{j i_2 \cdots i_N} \mathcal{B}^{k i_2 \cdots i_N} . \qquad (4)$$

As shown in [5] this extended Lagrangian includes that of $SU(3)$ in the case $N = N_f = 3$, for the 0^- meson–$1/2^+$ baryon sector.

Because the covariant derivative acts on each quark index in $\mathcal{B}^{i_1 \cdots i_N}$, there is an extra factor of N beyond that coming from the meson weak-decay constant. This would imply a large-N dependence of $\mathcal{O}(N^0)$ for the WT amplitude, of the same order as that of the p-wave amplitude [3]. This holds for generic baryons. The situation is more subtle if only baryons with finite spin and flavor in the large-N limit are retained [5]. In this case the spin-flavor extended WT amplitude is $\mathcal{O}(N^0)$ only if vector (and/or flavored) mesons are involved, but $\mathcal{O}(N^{-1})$ in the pseudoscalar (or flavorless) meson sector. This is the standard result. Indeed, the commutator of the absorbed and emitted meson fields in $\mathcal{L}^{\mathrm{sf}}_{\mathrm{WT}}$ just produces a spin-flavor generator to be measured by the baryonic matrix element. If the mesons carry only flavor but not spin, the generator will also be purely flavor-like, and flavor was finite by assumption. This gives a power of N less than for a generic baryon.

Note the group structure [5]

$$\mathcal{L}^{\mathrm{sf}}_{\mathrm{WT}} \sim ((M^\dagger \otimes M)\text{``}\mathbf{35}_A\text{''} \otimes (B^\dagger \otimes B)\text{``}\mathbf{35}\text{''})_1 \qquad (5)$$

(M, $M^\dagger$, B and $B^\dagger$ representing annihilation and creation operators for mesons and baryons) which indicates a purely antisymmetric adjoint coupling in the t-channel. As we will see below (cf. (7)), in the s-channel (or u-channel) there are four possible $SU(n)$ irreps, and so four independent couplings for a generic $SU(n)$ invariant meson-baryon Hamiltonian[1]. In the t-channel one has instead

$$\text{``}\mathbf{35}\text{''} \otimes \text{``}\mathbf{35}\text{''} = \mathbf{1} \oplus \text{``}\mathbf{35}_S\text{''} \oplus \text{``}\mathbf{35}_A\text{''} \oplus \text{``}\mathbf{405}\text{''} \oplus \cdots ,$$

$$\text{``}\mathbf{56}\text{''} \otimes \text{``}\mathbf{56}^*\text{''} = \mathbf{1} \oplus \text{``}\mathbf{35}\text{''} \oplus \text{``}\mathbf{405}\text{''} \oplus \cdots , \qquad (6)$$

where all the irreps not explicited differ from mesons to baryons. Thus there are again four independent coupling constants in the t-channels, as it should be. "$\mathbf{405}$" is the representation with tableau $[4, 2^{n-2}]$. The detailed group structure of WT comes as a consequence of the $SU(n)$ chiral-symmetry requirement [7].

Phenomenological consequences of the extended WT interaction in the negative-parity baryon sector are analyzed in [5].

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[1] This is the generic case. Some irreps may not exits for particular low values of n or N, see (8).

3 Crossed projectors

The coupling of baryons and mesons gives four $SU(n)$ irreps

$$\text{``}\mathbf{56}\text{''} \otimes \text{``}\mathbf{35}\text{''} = \text{``}\mathbf{56}\text{''} \oplus \text{``}\mathbf{70}\text{''} \oplus \text{``}\mathbf{700}\text{''} \oplus \text{``}\mathbf{1134}\text{''} \qquad (7)$$

with tableaus $[N]$, $[N - 1, 1]$, $[N + 2, 1^{n-2}]$ and $[N + 1, 2, 1^{n-3}]$, respectively. In what follows they will be called simply irreps 1, 2, 3, and 4. They have dimensions

$$d_1 = \binom{n + N - 1}{N} , \quad d_2 = \frac{(N - 1)(n - 1)}{n + N - 1} d_1,$$

$$d_3 = \frac{(n + N - 1)(n - 1)}{N + 1} d_1,$$

$$d_4 = \frac{nN(n + N)(n - 2)}{(n + N - 1)(N + 1)} d_1 . \qquad (8)$$

For convenience we introduce a vector notation so that $\boldsymbol{d} = [d_1, d_2, d_3, d_4]$.

An $SU(n)$ invariant Hamiltonian takes therefore the form (again using a vector/tensor notation in irrep space)

$$H = \sum_{\alpha=1}^4 h_\alpha P_\alpha = \boldsymbol{h} \cdot \boldsymbol{P}, \qquad (9)$$

where P_α denotes the projector of meson-baryon states on the irrep α. For the extended WT interaction, a direct calculation using Wick contractions [5] gives

$$\boldsymbol{h}_{\mathrm{WT}} = -\boldsymbol{\lambda}_{\mathrm{WT}} (\sqrt{s} - M)/f^2,$$

$$\boldsymbol{\lambda}_{\mathrm{WT}} = [n, N + n, -N, 1]. \qquad (10)$$

The WT amplitude is therefore $\mathcal{O}(N^0)$ for the irreps 2 and 3 and $\mathcal{O}(N^{-1})$ for 1 and 4. It is noteworthy the relation

$$\boldsymbol{\lambda}_{\mathrm{WT}} \cdot \boldsymbol{d} = 0, \qquad (11)$$

which embodies the property $\mathrm{tr}(H_{\mathrm{WT}}) = 0$, in turn coming from the vanishing of the trace of the meson operator living in the adjoint representation (obtained from the commutator of the meson fields).

As said the WT amplitude is purely "$\mathbf{35}$" in the t-channel and $\boldsymbol{\lambda}_{\mathrm{WT}}$ reflects this group structure as seen in the s-channel. In what follows we investigate how the various irrep in the t- and u-channels are seen in the s-channel. The u-channel corresponds to crossing of the meson legs. For instance a direct baryon-pole (s-channel) mechanism would give a pure "$\mathbf{56}$" coupling[2], i.e., $\boldsymbol{h}_{\mathrm{dbp}} \propto [1, 0, 0, 0]$, while a crossed baryon-pole (u-channel) mechanism

$$((M \otimes B^\dagger)\text{``}\mathbf{56}\text{''}_* \otimes (M^\dagger \otimes B)\text{``}\mathbf{56}\text{''})_1 \qquad (12)$$

gives instead (see Appendix D of [5a])

$$\boldsymbol{\lambda}_{\mathrm{cbp}} = \left[N + n - 1 - \frac{N}{n} - \frac{n}{N}, -1 - \frac{n}{N}, 1, -\frac{1}{N} \right] \qquad (13)$$

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[2] We assume here an hypothetical $SU(n)$ invariant meson-baryon-baryon coupling. The physical coupling is p-wave, hence the orbital momentum would have to be included.

Table 1. Matrix a.

$$
\mathsf{a} = \begin{bmatrix}
\dfrac{n(n+N)(N-1)-N^2}{N(n-1)(n+N)} & -\dfrac{n}{N(n-1)} & \dfrac{n}{(n-1)(n+N)} & -\dfrac{n}{N(n-1)(n+N)} \\[2ex]
-\dfrac{n(N-1)}{N(n+N-1)} & \dfrac{n}{N(n+N-1)} & \dfrac{N-1}{(n+N-1)} & -\dfrac{N-1}{N(n+N-1)} \\[2ex]
\dfrac{n(n+N+1)}{(N+1)(n+N)} & \dfrac{n+N+1}{(N+1)} & \dfrac{n}{(n+N)(N+1)} & \dfrac{n+N+1}{(n+N)(N+1)} \\[2ex]
-\dfrac{n^2(n-2)}{(n-1)(N+1)(n+N-1)} & -\dfrac{n(n-2)(n+N)}{(n-1)(N+1)(n+N-1)} & \dfrac{nN(n-2)}{(n-1)(N+1)(n+N-1)} & \dfrac{N(n-1)(n+N)+1}{(n-1)(N+1)(n+N-1)}
\end{bmatrix}
$$

(the normalization is arbitrary). While the effect of crossing for other irreps could be computed as in [5], by using Wick contractions, we present here a shortcut. To this end we introduce the crossed projectors

$$\langle m_1, b_1 | P_\alpha^c | m_2, b_2 \rangle = \langle \overline{m_2}, b_1 | P_\alpha | \overline{m_1}, b_2 \rangle, \qquad (14)$$

where $|\overline{m}\rangle = C|m\rangle$ indicates the crossed meson state.

Since the P_α^c themselves are invariant operators, we have

$$P_\alpha^c = \sum_\beta a_{\alpha\beta} P_\beta, \quad \text{or} \quad \boldsymbol{P}^c = \mathsf{a} \cdot \boldsymbol{P}, \qquad (15)$$

where the (real) crossing coefficients $a_{\alpha\beta}$ are those to be determined. They describe the u-channel irreps as seen in the s-channel.

For the group $SU(2)$ the (antilinear) operator C is just $|\overline{j,m}\rangle = (-1)^{j-m}|j,-m\rangle$, however, in the general case, the only property needed of C is to be a unitary conjugation, *i.e.*, $CC^\dagger = C^2 = 1$. Together with general projector properties

$$
\begin{aligned}
P_\alpha &= P_\alpha^\dagger, & P_\alpha P_\beta &= \delta_{\alpha\beta} P_\alpha, \\
\operatorname{tr} P_\alpha &= d_\alpha, & \sum_\alpha P_\alpha &= 1,
\end{aligned}
\qquad (16)
$$

one easily establishes the relations

$$
\begin{aligned}
\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} &= \delta_{\alpha\beta}, & \sum_\gamma a_{\alpha\gamma} a_{\beta\gamma} d_\gamma &= d_\alpha \delta_{\alpha\beta}, \\
\sum_\alpha a_{\alpha\beta} &= 1, & \sum_\beta a_{\alpha\beta} d_\beta &= d_\alpha.
\end{aligned}
\qquad (17)
$$

The first two relations imply that the matrix

$$\hat{a}_{\alpha\beta} = d_\alpha^{-1/2} a_{\alpha\beta} \, d_\beta^{1/2} \qquad (18)$$

satisfies $\hat{\mathsf{a}} = \hat{\mathsf{a}}^T = \hat{\mathsf{a}}^{-1}$, *i.e.*, it has four orthonormal eigenvectors $\hat{e}_\lambda$ with eigenvalues ± 1. As a consequence $\hat{\mathsf{a}}$ can be written as $P_{+1} - P_{-1} = 1 - 2P_{-1}$ ($P_{\pm 1}$ being the orthogonal projectors on the ± 1 subspaces of the four dimensional irrep space) and a is just $\hat{\mathsf{a}}$ expressed in a nonorthonormal basis. The third and fourth relations in (17) identify $[1, 1, 1, 1]$ as one such left eigenvector of a, with eigenvalue $+1$, and $\boldsymbol{d}$ as its right eigenvector version.

The left eigenvector $\boldsymbol{\lambda}_1 = [1, 1, 1, 1]$ can also be identified as the s-channel view of the t-channel coupling $((M^\dagger \otimes M)_{\mathbf{1}} \otimes (B^\dagger \otimes B)_{\mathbf{1}})_{\mathbf{1}}$ (this operator just counts

the number of mesons and baryons), and the eigenvalue $+1$ indicates that $\mathbf{1}$ comes from a symmetric coupling of "**35**" $\otimes$ "**35**".

On the other hand, the WT interaction (irrep "$\mathbf{35}_A$" in the t-channel) is antisymmetric under meson crossing and so $\boldsymbol{\lambda}_{\mathrm{WT}}$ is a left eigenvector of a with eigenvalue -1 (correspondingly, it is orthogonal to $\boldsymbol{\lambda}_1$, see (11).) The two other left eigenvectors correspond to "$\mathbf{35}_S$" and "$\mathbf{405}$" in the t-channel. Since they are both symmetric under crossing, they carry eigenvalue $+1$.

In view of the fact that $\boldsymbol{\lambda}_{\text{“}\mathbf{35}_A\text{”}} = \boldsymbol{\lambda}_{\mathrm{WT}}$ is the unique eigenvector with eigenvalue -1, we can directly reconstruct a as follows[3]

$$a_{\alpha\beta} = \delta_{\alpha\beta} - 2 \frac{\lambda_\alpha \lambda_\beta \, d_\alpha}{\sum_\gamma \lambda_\gamma^2 \, d_\gamma}. \qquad (19)$$

The matrix is displayed in table 1. Note that $\boldsymbol{\lambda}_{\mathrm{cbp}}$ is correctly reproduced (up to a factor) by the first row of a. This matrix contains all u-channel results. For the "$\mathbf{35}_S$" t-channel, a direct Wick computation gives

$$
\boldsymbol{\lambda}_{\text{“}\mathbf{35}_S\text{”}} = \\
\left[\left(\frac{2}{n} + \frac{1}{N} \right)(n-2), \left(\frac{1}{n} + \frac{1}{N} \right)(n-2), \frac{n-2}{n}, -\frac{1}{N} - \frac{2}{n} \right].
$$
$$(20)$$

As can be verified it is a $+1$ left eigenvector of a and is orthogonal to $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_{\text{“}\mathbf{35}_A''\text{”}}$. Finally, by orthogonality we obtain

$$\boldsymbol{\lambda}_{\text{“}\mathbf{405}\text{”}} = \left[n, -\frac{n+N}{N-1}, -\frac{N}{n+N+1}, \frac{1}{n} \right]. \qquad (21)$$

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[3] Since $\hat{\mathsf{a}} = 1 - 2P_{-1} = 1 - 2\hat{e}_{-1} \otimes \hat{e}_{-1}$.

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